

James Clerk Maxwell and the dynamics of astrophysical discs

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Maxwell's investigations into the stability of Saturn's rings provide one of the earliest analyses of the dynamics of astrophysical discs. Current research in planetary rings extends Maxwell's kinetic theory to treat dense granular gases of particles undergoing moderately frequent inelastic collisions. Rather than disrupting the rings, local instabilities may be responsible for generating their irregular radial structure. Accretion discs around black holes or compact stars consist of a plasma permeated by a tangled magnetic field and may be compared with laboratory fluids through an analogy that connects Maxwell's researches in electromagnetism and viscoelasticity. A common theme in this work is the appearance of a complex fluid with a dynamical constitutive equation relating the stress in the medium to the history of its deformation.

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1. Maxwell 150 years ago

In 1856, the year that James Clerk Maxwell left Trinity College, Cambridge to become Professor of Natural Philosophy at Marischal College, Aberdeen, he was also writing his Adams Prize essay *On the stability of the motion of Saturn's rings*. The prize, named after the astronomer John Couch Adams who predicted the existence of Neptune, was awarded every two years by St John's College and the University of Cambridge for a mathematical essay on a specified topic. Among the examiners for the 1856 essay with its planetary theme were James Challis and William Thomson, later Lord Kelvin. Challis was the Plumian Professor of Astronomy whose lack of interest in Adams's prediction led him famously to fail to discover Neptune.

Such was the perceived difficulty of the chosen subject that Maxwell's was the only entry in the competition. The essay topic required Maxwell to investigate the stability of various configurations of solid, liquid and particulate rings orbiting around Saturn. Laplace had already shown that a uniform solid ring would be unstable, and suggested that the rings were solid but of non-uniform density. Maxwell's essay is interesting partly for its use of Fourier analysis and dispersion relations in the determination of stability criteria. For example, he considered the behaviour of a circular ring of equally spaced identical satellites under the destabilizing influence of their mutual gravitational attraction. He did this by introducing a perturbation in the form of a displacement that was a periodic function of the azimuthal angle ϕ and would depend on time through a complex exponential function,

being proportional (in a modern notation) to $\exp(im\phi + st)$. An algebraic dispersion relation followed, which gave the growth rate s in terms of the wavenumber m . If a solution existed with a positive real part for any suitable value of m he deduced that the configuration was unstable and therefore untenable as a model of Saturn's rings. Maxwell's conclusion was that "the only system of rings which can exist is one composed of an indefinite number of unconnected particles, revolving around the planet with different velocities according to their respective distances". The following year he expressed himself somewhat more poetically in a letter to Thomson, describing the rings as "a great stratum of rubbish jostling and jumbling round Saturn without hope of rest or agreement in itself".

The published version of Maxwell's essay (Maxwell 1859) includes the results of work done in 1857 after the competition was closed. Brush, Everitt & Garber (1983) provide a comprehensive historical discussion together with reprints of all the relevant documents. A detailed scientific critique of Maxwell's essay has been given by Cook & Franklin (1964).

2. Modern view of Saturn's rings

The modern view of Saturn's rings (*e.g.* Esposito 2006 and references therein) benefits, of course, from space exploration, notably the *Voyager* encounters of the early 1980s and the current *Cassini* mission. The principal rings of Saturn are the outer A ring and the inner B ring, separated by the 4500-km Cassini division. Much of the structure in the A ring can be attributed to its gravitational interaction with some of the moons of Saturn. The narrow Encke and Keeler gaps are cleared by moonlets orbiting within the A ring, which also generate trailing wakes and wavy edges. Also seen in the A ring are many discrete orbital resonances with more massive moons orbiting outside the ring system; at these locations spiral density or bending waves are launched, which propagate some distance towards or away from Saturn before being damped. The B ring, however, contains a rich irregular radial structure that cannot be attributed to satellite resonances, and which remains one of the major puzzles of ring dynamics (*e.g.* Tremaine 2003).

Although even *Cassini* cannot resolve individual ring particles, modern observations concur with Maxwell's conclusion regarding the constitution of the rings. The system consists of a dense 'granular gas' of iceballs up to several metres in diameter, the size distribution being inferred from radio occultation experiments. Ring particles are in circular orbital motion in Saturn's equatorial plane, at velocities on the order of 10 km s^{-1} . In addition they have random velocities on the order of 1 mm s^{-1} . The tiny ratio of 10^{-7} is therefore characteristic of the orbital eccentricities and inclinations of ring particles, and also of the angular semithickness of the rings.

Owing to their velocity dispersion, ring particles experience collisions that are very gentle but nevertheless significantly inelastic because of the nature of the material. Particles undergo a few collisions per orbit, placing the system in an interesting but challenging physical regime. It is different on the one hand from the situation in stellar dynamics, where the probability of a star undergoing a significant scattering event with another star within one orbit around a galaxy is typically very small. It differs on the other hand from gas dynamics, where molecules usually experience very many elastic collisions within the time-scales characteristic of the

macroscopic flow and therefore establish a locally Maxwellian velocity distribution. The implication is that the rings do behave as a continuous medium similar to a fluid, but a complex or non-Newtonian one with a non-trivial rheology that needs to be explored. The velocity distribution can be significantly anisotropic.

Gravitational forces between ring particles are not negligible because the escape speed from the surface of a typical particle may be comparable to the velocity dispersion. As well as affecting the collisional dynamics to some extent, self-gravity as a collective effect makes the rings thinner and is especially important in the propagation of waves.

An important aspect of ring dynamics is the establishment of an energy equilibrium. Owing to Kepler's third law, the rings undergo systematic shear, with the inner parts rotating more rapidly than the outer. When particles collide their relative velocity results partly from their random velocities and partly from the systematic shear. Velocity dispersion can therefore be generated from shear in the same way that viscosity generates heat in a shearing fluid. On the other hand the inelasticity of the collisions diminishes the random velocities. The velocity dispersion is therefore set by the restitution law of the material. Experiments at UC Santa Cruz have determined how the coefficient of restitution of ice at low temperatures and pressures decreases with increasing collision speed (*e.g.* Hatzes, Bridges & Lin 1988). An energy equilibrium in the rings can be found when the typical collision speed is such as to imply a critical amount of inelasticity (Goldreich & Tremaine 1978). If the coefficient of restitution is always too small to allow this solution, another equilibrium may be found in which the velocity dispersion is comparable to the shear velocity across a particle diameter.

There are essentially two possible approaches to the theoretical study of Saturn's rings, N -body simulations and statistical theories, each of which has its merits and limitations. Direct N -body simulations (*e.g.* Salo, Schmidt & Spahn 2001) have the character of virtual laboratory experiments, and the accuracy of the model is constrained mainly by our uncertainty about the physical properties of the particles. An obvious limitation of this approach, however, lies in the limited scale of the simulations. For example, the number of metre-sized particles in Saturn's B ring is on the order of 10^{15} , whereas a large-scale numerical simulation involves about 10^5 particles. Simulations are therefore restricted to a tiny patch of a ring, corresponding to length-scales that are only just observable with *Cassini*; they are also limited to durations very much shorter than the global evolutionary time-scale of the ring. The extreme thinness of the rings is an obstacle to global simulations, because of the great ranges of length and time-scales involved. In order to understand the very direct and detailed information provided by simulations and to apply it to phenomena on larger scales or in different circumstances, the results need to be assimilated within a theoretical framework of some kind. Therefore it is important to develop analytical models in parallel with advances in computation.

A statistical approach to planetary ring dynamics is natural in view of the very large number of particles and the desire to formulate a continuum description of some kind. Nevertheless, the difficulties involved are severe. Such work builds on the foundations of kinetic theory laid by Maxwell. Indeed, the manuscripts held in Cambridge University Library (reprinted in Brush *et al.* 1983) show that Maxwell clearly conceived the idea of the energy equilibrium in Saturn's rings. He also started to develop a kinetic theory of the rings, but it proved much more difficult than

the theory of gases because of the inelastic nature of the collisions. Their relative infrequency also means that the velocity distribution is significantly anisotropic and therefore non-Maxwellian.

3. Kinetic theory of Saturn's rings

The aim of a statistical approach is to derive a continuum mechanical model of the rings, consisting of a set of partial differential equations related to, but more complicated than, the Navier–Stokes equations of fluid dynamics. To do this one may start from Enskog's equation governing the collisional evolution of the distribution function $f(\mathbf{x}, \mathbf{v}, t)$, which gives the number density of particles in the six-dimensional position–velocity phase space. This is similar to Boltzmann's equation in gas dynamics but allows for the dense nature of the medium. It is also necessary to modify Enskog's equation to allow for the inelasticity of the collisions (Jenkins & Richman 1985; Araki & Tremaine 1986).

In order to make analytical progress, it is supposed (in common with most N -body simulations) that the particles are identical, smooth, indestructible spheres. Collisions are assumed to involve only two particles at a time, and gravitational scattering or focusing is neglected.

The description can be reduced from six dimensions to three by taking velocity moments of Enskog's equation. The zeroth and first moments generate equations for the conservation of mass and momentum, while the second moment generates a dynamical equation for the kinetic stress or velocity dispersion tensor \mathbf{W} (equation 3.3 below). This can be regarded, in part, as the constitutive equation for the complex fluid. Rather than relating the stress instantaneously and locally to the rate of strain, as in a Newtonian viscous fluid, the dynamical constitutive equation accounts for the fact that the medium has a finite 'memory' of its deformation history. To close the moment hierarchy it is necessary to neglect, or otherwise model, the third moments that appear. The collision term in the second-moment equation can be evaluated by modelling the distribution function as a triaxial Gaussian. Experience suggests that this relatively crude method of treating the velocity dimensions in Enskog's equation can yield quite accurate results because of the smoothing effect of the integral operator in the collision term.

In addition to the kinetic stress there is also a nonlocal, or collisional, stress requiring further integrals. This contribution originates from the direct transfer of momentum between particles during a collision, rather than the carriage of momentum by particles between collisions, and can be dominant in a dense gas.

The result of this procedure is a closed system of nonlinear partial differential equations of the form (Latter 2006)

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0, \quad (3.1)$$

$$\frac{\partial}{\partial t}(mn\mathbf{u}) + \nabla \cdot (mn\mathbf{u}\mathbf{u} + mn\mathbf{W} + \mathbf{P}) = -mn\nabla\Phi, \quad (3.2)$$

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{W} + \mathbf{W} \cdot \nabla \mathbf{u} + (\nabla \mathbf{u})^T \cdot \mathbf{W} = \dot{\mathbf{W}}_c, \quad (3.3)$$

where m is the particle mass, n the number density, \mathbf{u} the bulk velocity, \mathbf{W} the velocity dispersion tensor and Φ the gravitational potential. The collisional stress

\mathbf{P} and the collisional rate of change of velocity dispersion, $\dot{\mathbf{W}}_c$, are complicated algebraic or numerical functions of n , $\nabla \mathbf{u}$ and \mathbf{W} .

Having set up this system of equations, one may apply it to a local model of a differentially rotating disc known as the shearing sheet. This uses the extreme thinness of the rings to replace the global cylindrical geometry with a local Cartesian one in which the systematic motion is represented as a linear shear flow in a uniformly rotating frame of reference. An approximate procedure of vertical averaging is used to treat the dimension perpendicular to the ring plane.

First, solutions may be sought that are independent of position and time, representing the local uniform statistical equilibrium states of the rings. Such equilibria incorporate the energy balance mentioned above, and predict a certain anisotropic shape for the velocity distribution. These solutions depend on the restitution law of the material and on various dimensionless parameters such as the normal optical thickness, which is an observable quantity and a measure of the areal density of ring material.

Following Maxwell's approach, one may then study the stability of the equilibria with respect to small disturbances with a sinusoidal dependence on (radial) position and a complex exponential dependence on time, being proportional to $\exp(ikx + st)$. It is possible to derive a complicated algebraic dispersion relation (with numerically determined coefficients) giving the growth rate s as a function of the wavenumber k . If any of the solutions has a positive real part, an instability is present.

Two types of instability that have been discussed on the basis of viscous (*i.e.* fluid dynamical) models of the rings are the viscous instability and the viscous overstability. The former is favoured when the viscosity decreases with surface density, the latter when it increases sufficiently rapidly. In each case the instability would occur for all wavelengths exceeding a critical value, although the growth rate diminishes rapidly with increasing wavelength. A kinetic treatment of dilute rings shows that the viscous overstability is in fact completely suppressed because of the non-Newtonian nature of the kinetic stress (Latter & Ogilvie 2006). However, the overstability is found to occur in dense rings where the collisional stress is important, in agreement with N -body simulations, if the optical thickness exceeds a critical value (Latter 2006).

In Maxwell's problem instability meant that the proposed rotating configuration could be ruled out as a model of Saturn's rings. However, the instabilities that occur in a kinetic model need not be destructive to the rings. In fact, their nonlinear evolution may lead to a saturated state, most likely an equilibrium of a statistical nature, involving regular or irregular structure and dynamics. Since the linear instabilities exist on a broad range of wavelengths, only the nonlinear dynamics can determine the statistical distribution of energy between the available scales. Future work may determine whether this behaviour can explain the rich irregular structure of Saturn's B ring.

4. The broader perspective

In astronomical terms, Saturn's rings are just one example of a universal phenomenon: the astrophysical disc. There are many situations in which a disc of fluid or solid matter is found in Keplerian orbital motion around a massive central body (*e.g.* Frank, King & Raine 2002). Protoplanetary discs, such as the 'solar nebula'

that once surrounded the Sun, occur as part of the process of star formation and contain most of the angular momentum of the cloud that collapsed to produce the star. These discs of dusty molecular gas last a few million years and are the sites of planet formation. High-energy accretion discs consisting of a fully ionized plasma are found around black holes or compact stars in interacting binary stars within the Galaxy, and also around much more massive black holes at the centres of active galaxies and quasars.

Despite the very different physical constitution of these various systems, they share common dynamical features. In each case a shear stress (often called a viscous stress) that develops in the differentially rotating flow transports angular momentum outwards. In accretion discs this allows inward mass transport towards the central body. One of the challenges of accretion disc theory is to understand and quantify the effects that give rise to this stress — that is, to understand the rheology of the disc.

While in planetary rings the stress arises from the kinetics of colliding particles, in high-energy accretion discs it is believed to result from turbulence in the presence of a magnetic field. The magnetorotational instability is a robust dynamical instability of a differentially rotating flow in which the angular velocity decreases outwards (Balbus & Hawley 1998). Rather than completely disrupting the orbital motion, it generates small-scale anisotropic magnetohydrodynamic turbulence in the disc, which transports angular momentum outwards. While the identification of the role of this instability represents a great advance in the theory of accretion discs, understanding the behaviour of the turbulent plasma is still a major challenge that has been attempted almost exclusively with direct numerical simulations of the magnetohydrodynamic equations in three dimensions.

5. A Maxwellian ‘mechanical model’

In the world of nineteenth-century physics it was often seen as desirable to develop mechanical models of new phenomena. Maxwell constructed a device involving ivory balls to illustrate the patterns of displacements in his theory of Saturn’s rings. In developing his great electromagnetic theory, he found it useful to formulate a mechanical model of ‘molecular vortices’ and ‘idle wheels’ (Maxwell 1861), even though he later adopted a more abstract Lagrangian approach (Maxwell 1865). Here I discuss a physical and mathematical analogy that links two quite different areas of Maxwell’s work and connects astrophysical fluid dynamics with the laboratory.

The great rivals Newton and Hooke both derived constitutive equations for different types of medium. While Newton proposed that the viscous stress in a fluid is proportional to the shear rate, or rate of strain, Hooke’s law stated that the elastic stress in a solid is proportional to the strain itself. In his paper *On the dynamical theory of gases* (1867) Maxwell wrote down an equation that interpolated between these descriptions and would later give rise to the theory of viscoelasticity:

$$\frac{d(\text{stress})}{dt} = \text{elasticity} \times \frac{d(\text{strain})}{dt} - \frac{\text{stress}}{\tau}.$$

According to Maxwell, a deformable medium has a characteristic relaxation time τ on which molecular interactions attempt to establish an isotropic velocity distribution and the shear stress, which is associated with anisotropy, decays. If a

deformation is applied that has a steady rate of strain or varies on a time-scale long compared to τ , the first term above is negligible and a viscous response occurs with the stress being proportional to the rate of strain as in Newton's law. However, if the strain varies on a time-scale short compared to τ , the third term above is negligible and an elastic response occurs with the stress being proportional to the strain as in Hooke's law. The reason for the elastic response is that the rapid strain prevents the molecules from relaxing towards an equilibrium distribution, and the stress is therefore 'frozen in' to the fluid during the deformation. In general Maxwell's equation allows a continuum of behaviour between viscous and elastic.

Modern constitutive equations for viscoelastic fluids (Bird, Armstrong & Hassager 1987a) are usually expressed in a covariant tensorial form based on the principles set out by Oldroyd (1950). His liquid B is one of the most widely used nonlinear models of a viscoelastic fluid, and provides an adequate representation of a dilute solution of a polymer of high molecular weight. It is based on Maxwell's viscoelastic equation above, but with a properly covariant interpretation of the time-derivatives. It can also be derived from the kinetic theory of idealized long-chain polymer molecules contained in a Newtonian solvent (Bird, Curtiss & Armstrong 1987b). If the viscosity of the solvent is negligible the model is known as the upper-convected Maxwell fluid. The dimensionless number characterizing the ratio of the relaxation time to the time-scale of the flow is the Deborah number; when this is large, the polymeric stress is effectively 'frozen in' to the fluid.

In an electrically conducting fluid such as the fully ionized plasma of an accretion disc around a black hole, the magnetic field \mathbf{B} affects the dynamics through the bulk Lorentz force (e.g. Roberts 1967). This can be represented in terms of the Maxwell electromagnetic stress tensor,

$$\mathbf{M} = \frac{\mathbf{B}\mathbf{B}}{\mu_0} - \frac{B^2}{2\mu_0}\mathbf{1}, \quad (5.1)$$

the two parts of which correspond to a tension in the field lines and an isotropic magnetic pressure (for non-relativistic flows, the electric field makes a negligible contribution to the stress). It is well known that, in a perfectly conducting fluid, the magnetic field is 'frozen in' to the fluid, in the sense that magnetic field lines can be identified with material lines (Alfvén 1950). Even in a fluid of finite conductivity, the magnetic field is effectively 'frozen in' for motions of sufficiently short time-scale, or sufficiently large length-scale, corresponding to a large magnetic Reynolds number. It follows that the Maxwell stress is also 'frozen in' to the fluid in a certain sense.

Ogilvie & Proctor (2003) have shown that there is a physical and mathematical similarity between the dynamics of viscoelastic and magnetohydrodynamic fluids. Either the polymer molecules or the magnetic field lines are advected and distorted by the flow and exert a tension force on it. A formal mathematical analogy can be drawn between the Oldroyd-B fluid in the limit of large Deborah number and a magnetized plasma in the limit of large magnetic Reynolds number. Either the polymeric tension or the magnetic tension in this limit satisfies the equation

$$\frac{\partial \mathbf{T}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{T} - \mathbf{T} \cdot \nabla \mathbf{u} - (\nabla \mathbf{u})^T \cdot \mathbf{T} = 0, \quad (5.2)$$

which states that the stress is ‘frozen in’ to the flow. This equation is also closely related to equation (3.3) for the kinetic stress in a planetary ring, although there is an important change of sign in two of the terms. The left-hand sides of these equations are both Lie derivatives, one for a contravariant tensor and the other for a covariant tensor.

Ogilvie & Proctor (2003) also showed that the magnetorotational instability, which, as mentioned above, is of primary importance in the theory of accretion discs, has a direct analogue in a viscoelastic fluid. Experiments have been carried out on the stability of viscoelastic Couette–Taylor flow between differentially rotating concentric cylinders since the 1960s. However, they appear to have neglected the regime in which the magnetorotational instability would appear most clearly; this requires that both cylinders rotate, but in such a way that the specific angular momentum increases outwards while the angular velocity decreases, as in a Keplerian disc. It is likely that an experiment using a dilute polymer solution could quite easily demonstrate the existence of this astrophysically important instability and determine its nonlinear evolution. In contrast, experimental efforts to demonstrate the instability in its magnetohydrodynamic version still face considerable technical difficulties (Goodman & Ji 2002).

6. Conclusion

Planetary rings provide the only examples of the universal phenomenon of astrophysical discs that can be observed with exquisite resolution. Investigations of the stability of Saturn’s rings 150 years after Maxwell’s still hold interest as they may explain the rich irregular structure in the B ring. Understanding the behaviour of planetary rings in detail has been difficult because of the complex physical nature of the medium: a non-Newtonian fluid consisting of a dense granular gas of particles undergoing inelastic and only moderately frequent collisions. Current research builds on the kinetic theories developed by Maxwell and others.

Other astrophysical discs also possess a complicated rheology owing to the presence of magnetohydrodynamic turbulence in a fully ionized plasma. A simple analogy with polymeric liquids may provide a way to understand aspects of their behaviour and even to study it experimentally. In all of this work the theory rests on the firm foundations established by Maxwell’s pioneering researches in dynamics, kinetic theory, electromagnetism and viscoelasticity.

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